Journal of Mathematical Sciences: Advances and Applications
Volume 18, Number 1-2, 2012, Pages 29-45

EXISTENCE OF HOLOMORPHICALLY FILLABLE CONTACT STRUCTURES ON SOME $\mathbb{T}^2$-BUNDLE OVER $\mathbb{S}^1$

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Abstract

The aim of this paper is to show that on some $\mathbb{T}^2$-bundle over $\mathbb{S}^1$, there exist a holomorphically fillable contact structure by relating different works of the authors of the papers [2], [4], and [6].

The main theorem is as follows:

Theorem 1. *Let M be a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with monodromy matrix $A \in SL_2(\mathbb{Z})$. If one of the following conditions holds*:

(1) $\operatorname{tr}(A) > 2$.

(2) *A is not periodic and satisfies* $|\operatorname{tr}(A)| = 2$.

(3) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$,

then there exist a holomorphically fillable contact structure on M.

2010 Mathematics Subject Classification: 53D05; 53D10.

Keywords and phrases: Sasakian contact metric structure, isomorphic Sasakian manifold, complex manifold, Sasakian structure, 3-dimensional compact manifold, holomorphically fillable contact structure.

Received September 22, 2012

1. Sasakian Structures

A contact form on a $2n-1$-dimensional manifold M is a 1-form α such that the identity

$$\alpha \wedge (d\alpha)^n \neq 0,$$

holds everywhere on M. Given such a 1-form α, there is always a unique vector field Z satisfying

$$\alpha(Z) = 1 \quad \text{and} \quad i_Z d\alpha = 0.$$

The vector field Z is called the *Reeb vector field* of the contact manifold (M, α) and the corresponding 1-dimensional foliation is called a *contact flow*. The $2n$-dimensional distribution

$$\xi(p) = \{v \in T_pM : \alpha(p)(v) = 0\},$$

which is invariant by Z, is called *the contact distribution*. It carries a (1, 1) tensor field J such that $-J^2$ is the identity on ξ. The tensor field J extends to all of TM, if one requires $JZ = 0$.

Also, the contact manifold (M, α) carries a nonunique Riemannian metric g called a *contact metric*, adapted to α and J in the sense that the following identities are satisfied for any vector fields X and Y on M.

$$g(X, Y) = g(JX, JY) + \alpha(X)\alpha(Y), \tag{1}$$

$$d\alpha(X, Y) = 2g(X, JY), \tag{2}$$

$$J^2X = -X + \alpha(X)Z, \quad JZ = 0. \tag{3}$$

Here, $d\alpha(X, Y)$ stands for

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]).$$

Moreover, since $0 = d\alpha(Z, X) = Z\alpha(X) - \alpha([Z, X])$, then for any X, one has

$$Z\alpha(X) = \alpha([Z, X]). \tag{4}$$

Lemma 1.1. *Let* (M, α, Z) *be a contact manifold with Reeb vector field* Z, *then for every tangent vector fields* X *and* Y *on* M, *we have*

$$Zd\alpha(X, Y) = d\alpha([Z, X], Y) + d\alpha(X, [Z, Y]).$$

Proof. With a direct calculation and the use of (4), one has

$$\begin{aligned} d\alpha([Z, X], Y) + d\alpha(X, [Z, Y]) \\ &= [Z, X]\alpha(Y) - Y\alpha([Z, X]) - \alpha([[Z, X], Y]) \\ &\quad - [Z, Y]\alpha[X] + X\alpha([Z, Y]) - \alpha([[Y, Z], X]) \\ &= ZX\alpha(Y) - ZY\alpha(X) - \alpha([Z, [X, Y]]) \\ &= Z(X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])) = Zd\alpha(X, Y). \end{aligned}$$ □

It is clear that $L_Z\alpha = 0$ and $L_Z d\alpha = 0$, but $L_Z J$ and $L_Z g$ need not vanish!

Proposition 1.2. *On a contact metric manifold* (M, α, J, g), $L_Z J = 0$ *if and only if* $L_Z g = 0$.

Proof. Indeed, for every tangent vector fields X and Y on M, we have

$$\begin{aligned} L_Z g(X, JY) &= Zg(X, JY) - g([Z, X], JY) - g(X, [Z, JY]) \\ &= Zg(X, JY) - g([Z, X], JY) - g(X, (L_Z J)Y) - g(X, J[Z, Y]) \\ &= \frac{1}{2} Zd\alpha(X, Y) - g([Z, X], JY) - g(X, (L_Z J)Y) - g(X, J[Z, Y]), \end{aligned}$$

and it follows from the Lemma 1.1 and Equation (2) that

$$\begin{aligned} L_Z g(X, JY) = \frac{1}{2}(d\alpha([Z, X], Y) + d\alpha(X, [Z, Y])) - \frac{1}{2} d\alpha([Z, X], Y) \\ - g(X, (L_Z J)Y) - \frac{1}{2} d\alpha(X, [Z, Y]), \end{aligned}$$

that is,

$$L_Z g(X, JY) = -g(X, (L_Z J)Y). \tag{5}$$

Therefore, if $L_Z g = 0$, then $(L_Z J)Y = 0$ for arbitrary Y.

Conversely, suppose $L_Z J = 0$. Then, from the above observation (5), $L_Z g(X, JY) = 0$ for any Y. We need to show that $L_Z g(X, Z) = 0$ for all X.

$$\begin{aligned} L_Z g(X, Z) &= Zg(X, Z) - g([Z, X], Z) \\ &= Z\alpha(X) - g([Z, X], Z) \\ &= \alpha([Z, X]) - \alpha([Z, X]) = 0. \end{aligned}$$

□

Definition 1.3. A contact metric structure on which $L_Z J = 0$ is called a *K-contact structure*.

Given a contact metric structure (M, α, Z, J, g), consider the product manifold $M \times \mathbb{R}$. A vector field on $M \times \mathbb{R}$ can be written as $X + f\frac{d}{dt}$, where X is tangent to M, t is the co-ordinate on $\mathbb{R}$, and f is a smooth function on $M \times \mathbb{R}$. We define an almost complex structure ϕ on $M \times \mathbb{R}$ by

$$\phi(X + f\frac{d}{dt}) = JX - fZ + \alpha(X)\frac{d}{dt}.$$

Definition 1.4. If ϕ is a complex structure, then we say that the contact structure (α, Z, J) is normal and the corresponding contact metric structure is called *Sasakian*.

By a classic theorem of Newlander and Nirenberg, an almost complex structure ϕ of class C^1 is a complex structure, if and only if its Nijenhuis torsion $[\phi, \phi]$ vanishes. The Nijenhuis torsion $[T, T]$ of a tensor field T of type (1, 1) is a tensor field given for any vector fields X, Y by

$$[T, T](X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY].$$

It was proven in the classical book of Blair [1] or in a recent paper of Rukimbira [9] that

Theorem 1.5 ([1], [9]). *A contact structure* (α, Z, J) *is normal*, *if and only if*

$$[J, J](X, Y) + d\alpha(X, Y)Z = 0, \tag{6}$$

$$JX\alpha(Y) - JY\alpha(X) - \alpha([JX, Y]) - \alpha([X, JY]) = 0, \tag{7}$$

$$(L_Z J)X = 0. \tag{8}$$

It follows from the above Theorem 1.5 that a Sasakian contact metric structure is K-contact. And in [9], Rukimbira has shown that the converse holds in dimension 3, that is:

Proposition 1.6 ([9]). *A K-contact* 3-*dimensional manifold is Sasakian*.

Definition 1.7. Let (M, α, Z, J, g) be a Sasakian manifold and let Φ be a diffeomorphism on M, then $(M, \Phi^*\alpha, \Phi_*^{-1}Z, \Phi_*^{-1}J\Phi_*, \Phi^*g)$ is called an *isomorphic Sasakian manifold*.

2. Strictly Plurisubharmonic Functions and Holomorphic Fillability

2.1. Contact structures coming from complex geometry

Let $\mathbb{X}$ be a connected complex manifold of complex dimension $n \geq 2$ and M be a real smooth hypersurface in $\mathbb{X}$. Denote by $J : T\mathbb{X} \to T\mathbb{X}$ the (integrable) almost complex structure associated to the complex structure of $\mathbb{X}$, where $T\mathbb{X}$ denotes the tangent bundle of the underlying smooth manifold of $\mathbb{X}$. Then $J(TM)$ cannot be equal to TM, because this last space is odd-dimensional. Therefore,

$$\xi := TM \cap J(TM),$$

is a J-invariant subspace of real codimension 1 of TM, that is, a hyperplane distribution with a natural complex structure $J|_{\xi}$. We will call it the complex distribution of $M \hookrightarrow \mathbb{X}$.

For various hypersurfaces M, one can get all the degrees of integrability of this distribution, from the completely integrable case till the completely non-integrable (or contact) one. A general situation when ξ is automatically contact is obtained when M is strongly pseudoconvex.

Definition 2.1.1. Let ρ be a smooth function on $\mathbb{X}$. It is called *strictly plurisubharmonic* (abbreviated *spsh*), if

$$-d(\boldsymbol{d}^c \rho) > 0, \quad \text{where} \quad \boldsymbol{d}^c \rho := d\rho \circ J \in T^* \mathbb{X},$$

or equivalently, on each local complex chart $(U, z_1, \dots, z_n)$, the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)(p)$ is positive definite at all $p \in U$.

Example 2.1.2. Let $\mathbb{X} = \mathbb{C}^n \simeq \mathbb{R}^{2n}$, with complex co-ordinates $(z_1, \dots, z_n)$ and corresponding real co-ordinates $(x_1, y_1, \dots, x_n, y_n)$ via $z_j = x_j + iy_j$. Let

$$\rho_0(x_1, y_1, \dots, x_n, y_n) = \sum_{j=1}^{n} (x_j^2 + y_j^2) = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} z_j \bar{z}_j,$$

where $\bar{z}_j$ is the conjugate of z_j. Then, we have

$$\frac{\partial^2 \rho_0}{\partial z_j \partial \bar{z}_k} = \frac{\partial}{\partial z_j}\left(\frac{\partial \rho_0}{\partial \bar{z}_k}\right)(p) = \frac{\partial}{\partial z_j}(z_k) = \delta_{jk}.$$

Thus, for all $p \in \mathbb{C}^n$,

$$\left(\frac{\partial^2 \rho_0}{\partial z_j \partial \bar{z}_k}\right)(p) = (\delta_{jk}) = Id >> 0.$$

Then ρ_0 is spsh $\mathbb{C}^n$. □

Lemma 2.1.3. *Let* $\mathbb{X}$ *be a complex manifold,* ρ *be a smooth spsh function on* $\mathbb{X}$, M *be a complex submanifold, and* $i : M \hookrightarrow \mathbb{X}$ *be the inclusion map. Then* $i^* \rho$ *is spsh.*

Proof. Let $\dim_{\mathbb{C}} \mathbb{X} = n$ and $\dim_{\mathbb{C}} \mathbb{M} = n - m$. For $p \in M$, choose a chart $(U, z_1, \dots, z_n)$ for $\mathbb{X}$ centered at p and adapted to M, that is,

$$M \cap U = \{(z_1, \dots, z_n) \in \mathbb{X} / z_1 = \dots = z_m = 0\}.$$

In this chart, $i^*\rho(z_1, \dots, z_n) = \rho(0, 0, \dots, 0, z_{m+1}, \dots, z_n)$. Thus $i^*\rho$ is spsh, if and only if

$$\left(\frac{\partial^2 \rho}{\partial z_{m+j} \partial \bar{z}_{m+k}}\right)(0, \dots, 0, z_{m+1}, \dots, z_n),$$

is positive definite, which holds since this is a minor of

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)(0, \dots, 0, z_{m+1}, \dots, z_n). \qquad \square$$

Definition 2.1.4. If a co-oriented real hypersurface of $\mathbb{X}$ may be defined locally in the neighbourhood of any of its points as a regular level of a spsh function, which grows from its negative to its positive side, then it is called *strongly pseudoconvex*.

It is important to take care of the co-orientation of the hypersurface: seen from one side it is pseudoconvex, from the other it is pseudoconcave. The terminology was chosen such that the positive side is the pseudoconcave one, distinguished by the fact that holomorphic curves tangent to the hypersurface are locally contained in that side.

The announced general family of contact manifolds given by complex geometry is presented in the next proposition:

Proposition 2.1.5 ([8]). *The complex distribution of any strongly pseudoconvex hypersurface of a complex manifold is a* (*naturally oriented*) *contact structure.*

The simplest example of this type of construction is given by

Example 2.1.6. Let $\mathbb{X} = \mathbb{C}^n$, with $n \geq 2$ and $\rho_0 = \sum_{j=1}^{n} |z_j|^2$ as in the Example 2.1.2 above. This is a proper spsh function on $\mathbb{C}^n$. It follows from Lemma 2.1.3 and Proposition 2.1.5, that the complex distribution on any euclidean sphere centered at the origin is therefore a contact structure. Since homotheties centered at the origin leave both the foliation of $\mathbb{C}^n - \{0\}$ by such spheres and the almost complex structure invariant, they realize contactomorphisms between all such contact spheres. Therefore, one gets a well-defined contact structure on $\mathbb{S}^{2n-1}$, called the *standard contact structure* on it.

As in [3], we give several other examples, namely:

The contact boundary associated with a holomorphic immersion

Let $(\mathbb{X}, x)$ be an irreducible germ of a complex space with isolated singularity. Let $\mathcal{I}_{\mathbb{X}, x} \subseteq \mathcal{O}_{\mathbb{X}, x}$ be the maximal ideal of germs of holomorphic functions on $(\mathbb{X}, x)$ vanishing at x. Write $\mathbb{X}^*$ for the complex manifold $\mathbb{X} - \{x\}$.

Let $J : T\mathbb{X}^* \to T\mathbb{X}^*$ be the operator of fiberwise multiplication by i, when $T\mathbb{X}^*$ is seen as a real vector bundle. For any $\varphi_1, \ldots, \varphi_N \in \mathcal{I}_{\mathbb{X}, x}$, consider the holomorphic map

$$\Psi : (\mathbb{X}, x) \to (\mathbb{C}^N, 0),$$

with components φ_i, and the real analytic function

$$\gamma := \sum_{j=1}^{N} |\varphi_j|^2 : (\mathbb{X}, x) \to (\mathbb{R}, 0).$$

For $\varepsilon > 0$, define

$$M_{\gamma, \varepsilon} := \gamma^{-1}(\varepsilon).$$

Following [3], for $\varepsilon > 0$ sufficiently small, $M_{\gamma, \varepsilon}$, is a smooth compact 3-manifold, called a *link of* $(\mathbb{X}, x)$, if and only if Ψ is a finite analytic morphism. In the sequel, we will assume that this fact holds.

On $\mathbb{X}^*$, we consider the following natural objects associated with it:

$$\alpha := -\mathbf{d}^c\gamma.$$

Then on $\mathbb{X}^*$, define

$$\xi_\gamma := \mathrm{Ker}(d\gamma) \cap \mathrm{Ker}(\mathbf{d}^c\gamma).$$

It is a field of complex tangent hyper planes of the real tangent bundle of $\mathbb{X}^*$ endowed with its canonical (almost) complex structure. Moreover, it is tangent to the levels $M_{\gamma, \varepsilon}$, of γ. In fact,

$$\xi_{\gamma, \varepsilon} := \xi_\gamma | M_{\gamma, \varepsilon} = \mathrm{Ker}(\alpha | M_{\gamma, \varepsilon}).$$

And in [3], the following proposition is proven.

Proposition 2.1.7 ([3]). *The following conditions are equivalent*:

(1) *The pair* $(M_{\gamma, \varepsilon}, \xi_{\gamma, \varepsilon})$ *is a contact manifold for* ε *sufficiently small.*

(2) *The morphism* Ψ *is an immersion of* $\mathbb{X}^*$ *into* $\mathbb{C}^N$.

(3) *The function* γ *is spsh.*

Moreover, we know also from [3] that

Proposition 2.1.8 ([3]). *The pair* $(M_{\gamma, \varepsilon}, \xi_{\gamma, \varepsilon})$ *is a positive contact manifold*, *whose contact isotopy type does not depend on the choice of the holomorphic immersion* Ψ *and of* $\varepsilon > 0$ *sufficiently small. This isotopy type is called the contact boundary of* $(\mathbb{X}, x)$ *and denoted by* $(M(\mathbb{X}), \xi(\mathbb{X}))$.

2.2. Holomorphically fillable contact structures

Definition 2.2.1. A contact manifold, which is contactomorphic to the complex distribution on a strongly pseudoconvex boundary of a compact complex manifold with boundary is called *holomorphically fillable*.

In [4], Eliashberg prove the following proposition:

Proposition 2.2.2 ([4]). *On* $\mathbb{T}^3$, *the standard contact structure is up to homotopy, the unique holomorphically fillable one.*

One can find again holomorphically fillable contact structures by the following lemma:

Lemma 2.2.3. *The contact boundary* $(M(\mathbb{X}), \xi(\mathbb{X}))$ *of* $(\mathbb{X}, x)$ *as in the previous section is holomorphically fillable.*

Proof. Indeed, it is a direct consequence of the Propositions 2.1.7 and 2.1.8 and the Definitions 2.2.1 and 2.1.4. □

The following example will be used later, in order to prove existence of holomorphically fillable contact structures, on torus bundles over the circle with hyperbolic monodromy.

Example 2.2.4. The algebraic surface V in $\mathbb{C}^3$ defined by $V = f^{-1}\{0\}$, where

$$f(x, y, z) = x^p + y^q + z^r + xyz,$$

with $p, q, r \in \mathbb{Z}_{\geq 2}$ satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, has the only singularity at $0 = (0, 0, 0)$. For ε sufficiently small, set $K = V \cap \mathbb{S}^5_\varepsilon$, where $\mathbb{S}^5_\varepsilon$ is a small sphere centered at the origin. Let us consider in V the functions

$$\varphi_1(x, y, z) = x, \quad \varphi_2(x, y, z) = y, \quad \varphi_3(x, y, z) = z.$$

And

$$\gamma_0 := \sum_{j=1}^{3} |\varphi_j|^2 : (V, 0) \to (\mathbb{R}, 0).$$

γ_0 is the restriction on V of the map $\rho_0 = |x|^2 + |y|^2 + |z|^2$ defined in $\mathbb{C}^3$ as in Example 2.1.2. So γ_0 is spsh. Therefore, in virtue of Proposition 2.1.7, the pair $(M_{\gamma_0, \varepsilon^2}, \xi_{\gamma_0, \varepsilon^2})$ is a contact manifold. Moreover, it is easily seen that

$$K = \gamma_0^{-1}(\varepsilon^2) = M_{\gamma_0, \varepsilon^2},$$

and $\xi_{\gamma_0, \varepsilon^2}$ is exactly the restriction on K of the standard contact structure ξ_0 of $\mathbb{S}^5$.

Thus, $(K, \xi_0|_K)$ is isotopic to the contact boundary $(M(V), \xi(V))$ of $(V, 0)$ and then holomorphically fillable by Lemma 2.2.3 above. □

Holomorphically fillable contact structure can also be found in the world of Sasakian manifolds by the following result of Marinescu and Yeganefar in [7]:

Theorem 2.2.5 ([7]). *Every compact Sasakian manifold is holomorphically fillable.*

3. Prove of the Main Theorem

3.1. Compact quotient of the 3-Heisenberg group by discrete co-compact subgroup are holomorphically fillable

The 3-dimensional Heisenberg group Nil^3 can be described by the group of 3 by 3 real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

As a manifold it is just $\mathbb{R}^3$ with the following multiplication:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

Let $\mathcal{H}$ be the Lie algebra of Nil^3. For all $p = (x, y, z) \in Nil^3$, the left translation L_p satisfies

$$(dL_p)_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix},$$

where $e = (0, 0, 0) \in Nil^3$ is the unit element of the group. Every left invariant vector field

$$X = A(x, y, z)\frac{\partial}{\partial x} + B(x, y, z)\frac{\partial}{\partial y} + C(x, y, z)\frac{\partial}{\partial z},$$

is completely determinated by its value $X_e = (A_0, B_0, C_0)$ at e and for all $p \in Nil^3$, we have

$$X_p = (dL_p)_e X_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ B_0 x + C_0 \end{pmatrix}.$$

This mean that

$$X = A_0 \frac{\partial}{\partial x} + B_0\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right) + C_0 \frac{\partial}{\partial z}.$$

Therefore, the Lie group Nil^3 is parallelizable by the global vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z},$$

and the global left invariant 1-forms

$$dx,\ dz - xdy,\ dy.$$

Lemma 3.1.1. *The left invariant* 1-*form* $\alpha_L = dz - xdy$ *define a Sasakian structure on* Nil^3.

Proof. A direct calculation give

$$\alpha_L \wedge d\alpha_L = -dx \wedge dy \wedge dz \neq 0,$$

which mean that α_L is a contact form with $Z_L = \frac{\partial}{\partial z}$ as Reeb vector field. In order to prove that its associated contact structure is Sasakian, it is enough to see that $(Nil^3, \alpha_L, Z_L, J_L, g_L)$ is a K-contact 3-dimensional manifold, where

$$J_L = \left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right) \otimes dx - \frac{\partial}{\partial x} \otimes dy, \quad g_L = (dx)^2 + (dy)^2 + (\alpha_L)^2.$$

Let $X = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}$ and $Y = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}$ be two vector fields on Nil^3. Then, we see that

$$J_L X = -A_2\frac{\partial}{\partial x} + A_1\frac{\partial}{\partial y} + A_1 x\frac{\partial}{\partial z} \quad \text{and} \quad J_L Y = -B_2\frac{\partial}{\partial x} + B_1\frac{\partial}{\partial y} + B_1 x\frac{\partial}{\partial z}.$$

This mean that $L_{Z_L} g_L(X, Y) = 0$. And it follows from the Definition 1.3 and the Proposition 1.6 that $(Nil^3, \alpha_L, Z_L, J_L, g_L)$ is Sasakian.

□

Since α_L, Z_L, J_L, g_L are left invariant, then $S_L = (\alpha_L, Z_L, J_L, g_L)$ is the so-called left invariant Sasakain structure on Nil^3. Moreover, considering the involution map

$$\iota : \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x, y, z) \to (y, x, z),$$

we can in the same way determine a right invariant Sasakian structure, namely, $S_R = (\alpha_R, Z_R, J_R, g_R)$ on Nil^3, related to S_L by

$$\iota^*\alpha_L = \alpha_R,\ Z_L = \iota_*^{-1}Z_L = Z_R,\ \iota^* g_L = g_R,\ \iota_*^{-1}J_L\iota_* = J_R.$$

Thus, it follows from the Definition 1.7, that S_L and S_R are isomorphic. Then the Heisenberg group carry, what we call a bi-Sasakian structure, we can fix one of them and refer to it as the standard Sasakian structure on Nil^3. Moreover, it is proved in [2] that:

Theorem 3.1.2 ([2])**.** *Let M be a 3-dimensional compact manifold, which is diffeomorphic to a left quotient of the 3-dimensional Heisenberg group Nil^3, then the only Sasakian structure that passes down to the quotient is the standard one.*

So, it follows from the Theorems 3.1.2 and 2.2.5 that

Proposition 3.1.3. *3-dimensional compact manifolds, which are diffeomorphic to a left quotient of Nil^3 are holomorphically fillable.*

3.2. Sol-manifolds are holomorphically fillable

Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be the co-ordinate on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and $(\begin{pmatrix} x \\ y \end{pmatrix}, z)$ be the co-ordinate on $T^2 \times [0, 1]$. Let A be an element of $SL_2(\mathbb{Z})$ such that $\mathrm{tr}(A) > 2$. Then A has two positive eigenvalues $\lambda > 1$ and λ^{-1}, with corresponding eigenvectors E_+ and E_- satisfying $dx \wedge dy(E_+, E_-) = 1$.

Definition 3.2.1. We define an equivalence relation $\sim$ on $T^2 \times [0, 1]$ by identification of the two points $(A\begin{pmatrix} x \\ y \end{pmatrix}, z)$ and $(\begin{pmatrix} x \\ y \end{pmatrix}, z + 1)$. The quotient manifold $T_A = T^2 \times [0, 1] / \sim$ is called a *hyperbolic mapping torus*.

The Lie group Sol^3 is the split extension $1 \to \mathbb{R}^2 \to Sol^3 \to \mathbb{R} \to 1$, whose group structure is given on $\mathbb{R}^2 \times \mathbb{R}$ by

$$(u, v, w) \cdot (u', v', w') = (u + e^w u', v + e^{-w} v', w + w').$$

It is well know that T_A is equivalent to a Sol-manifold, namely, $\Gamma \setminus Sol^3$, a compact left quotient of Sol^3, where Γ is a co-compact discrete subgroup of Sol^3. Moreover, the left invariant 1-forms $\widetilde{\beta}_+ = e^{-w}du$ and $\widetilde{\beta}_- = -e^{w}dv$ on Sol^3 induce the 1-forms $\beta_+ = \lambda^{-z}dx \wedge dy(E_+, .)$ and $\beta_- = -\lambda^{z}dx \wedge dy(E_-, .)$ on T_A (for more details, see [6]).

Lemma 3.2.2. $\beta_+ + \beta_-$ *is a positive contact form on* T_A.

Proof. Indeed, we have

$$\begin{aligned}(\widetilde{\beta}_+ + \widetilde{\beta}_-) \wedge d(\widetilde{\beta}_+ + \widetilde{\beta}_-) &= (e^{-w}du - e^{w}dv) \wedge d(e^{-w}du - e^{w}dv) \\ &= (e^{-w}du - e^{w}dv) \wedge (e^{-w}du \wedge dw + e^{w}dv \wedge dw) \\ &= 2du \wedge dv \wedge dw > 0.\end{aligned}$$

Thus $\widetilde{\beta}_+ + \widetilde{\beta}_-$ is a positive left invariant contact form on Sol^3, this allows us to tell that the induced 1-form $\beta_+ + \beta_-$ is a positive contact form on T_A. □

Let us consider again $(K, \xi_0|_K)$ the holomorphically fillable contact manifold in the Example 2.2.4 above. One know from Kasuya in [6], that:

Proposition 3.2.3 ([6]). $(T_A, \beta_+ + \beta_-)$ *is contactomorphic to the link* $(K, \xi_0|_K)$, *where*

$$A = \left\{\begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & q \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & r \end{pmatrix}\right\}^{-1}. \tag{9}$$

This allows us to give the following proposition:

Proposition 3.2.4. $(T_A, \beta_+ + \beta_-)$ *is holomorphically fillable.*

Proof. As in [6], following Hirzebruch constructions, there exist always a basis in which the matrix A can be written like (9). Therefore, it follows from the Example 2.2.4 and the Proposition 3.2.3 that $(T_A, \beta_+ + \beta_-)$ is holomorphically fillable. □

3.3. Proof of the main Theorem 1

Theorem 3.3.1. *Let M be a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with monodromy matrix $A \in SL_2(\mathbb{Z})$. If one of the following conditions holds*:

(1) $\operatorname{tr}(A) > 2$.

(2) *A is not periodic and satisfies* $|\operatorname{tr}(A)| = 2$.

(3) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

then there exist a holomorphically fillable contact structure on M.

Proof. Let M be a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with monodromy matrix $A \in SL_2(\mathbb{Z})$. Following Geiges and Gonzalo in [5], we have

(1) If A satisfies $\operatorname{tr}(A) > 2$, the manifold M is a compact left quotient of Sol^3. Thus M looks like T_A and it follows from Proposition 3.2.3, that M is holomorphically fillable.

(2) If A is not periodic and satisfies $|\operatorname{tr}(A)| = 2$, M is a compact left quotient of Nil^3. The manifold M is so holomorphically fillable in virtue of Proposition 3.1.3.

(3) If the matrix $A = I_2$, M is diffeomorphic to the 3-torus $\mathbb{T}^3$, which contains a unique holomorphically fillable contact structure, as proven in Proposition 2.2.2. □

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